



## EE 232 Lightwave Devices Lecture 8: Einstein's AB Coefficients, Spontaneous Emission

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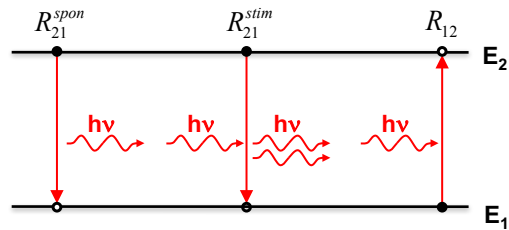
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### Einstein's AB Coefficients

$$\begin{aligned} R_{21}^{spont} &= A_{21} f_2 (1 - f_1) \\ R_{21}^{stim} &= B_{21} f_2 (1 - f_1) P(E_{21}) \\ R_{12} &= B_{12} f_1 (1 - f_2) P(E_{21}) \end{aligned}$$



For non-monochromatic light:

$$P(E_{21}) = n_{ph} N(E_{21}) :$$

number of photons per unit volume per energy interval

$$n_{ph} = \frac{1}{e^{h\omega_s/k_B T} - 1} : \text{Number of photons per state (Bose-Einstein distribution)}$$

$$N(E_{21}) = \frac{8\pi n_r^3 E_{21}^2}{h^3 c^3} : \text{Number of states with photon energy } E_{ba} \text{ per unit volume, per energy interval}$$

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## Photon Density of States

Optical wave  $e^{i\vec{k}\cdot\vec{r}}$  satisfies periodic boundary condition

$$\omega_k = \frac{kc}{n_r} \quad \text{dispersion relation of photons}$$

(equivalent to energy band structure of electrons)

Number of states with photon energy  $E_{21}$  per unit volume, per energy interval

$$N(E_{21}) = \frac{2}{V} \int \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \cdot \delta(E_{21} - \hbar\omega_k)$$

$$= \frac{8\pi}{(2\pi)^3} \int \left(\frac{n_r \omega_k}{c}\right)^2 \frac{n_r}{c} d\omega_k \cdot \frac{1}{\hbar} \delta\left(\frac{E_{21}}{\hbar} - \omega_k\right)$$

$$N(E_{21}) = \frac{8\pi n_r^3 E_{21}^2}{h^3 c^3}$$

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## Einstein's AB Coefficients

At thermal equilibrium:

$$R_{12} = R_{21}^{spon} + R_{21}^{stim}$$

$$B_{12} f_1 (1 - f_2) P(E_{21}) = A_{21} f_2 (1 - f_1) + B_{21} f_2 (1 - f_1) P(E_{21})$$

$$P(E_{21}) = \frac{A_{21} f_2 (1 - f_1)}{B_{12} f_1 (1 - f_2) - B_{21} f_2 (1 - f_1)} = \frac{A_{21} e^{\frac{E_1 - F}{k_B T}}}{B_{12} e^{\frac{E_2 - F}{k_B T}} - B_{21} e^{\frac{E_1 - F}{k_B T}}}$$

$$N(E_{21}) \cdot n_{ph} = \frac{A_{21}}{B_{12} e^{\frac{E_2 - E_1}{k_B T}} - B_{21}} \Rightarrow \left( \frac{8\pi n_r^3 E_{21}^2}{h^3 c^3} \right) \frac{1}{e^{\hbar\omega_k/k_B T} - 1} = \frac{A_{21}}{B_{12} e^{\frac{E_2 - E_1}{k_B T}} - B_{21}}$$

$$B_{12} = B_{21}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi n_r^3 E_{21}^2}{h^3 c^3} = N(E_{21})$$

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# Spontaneous Emission Spectra

$$B_{12} = B_{21} = B$$

$$\frac{A_{21}}{B} = \frac{8\pi n_r^3 E_{21}^2}{h^3 c^3} = N(E_{21})$$

$$R_{21}^{spont} = r_{21}^{spont}(E_{21})dE = A_{21}f_2(1-f_1)$$

$$R_{net}^{abs} = r_{net}^{abs}(E_{21})dE = B[f_2 - f_1]P(E_{21})$$

Absorption coefficient:

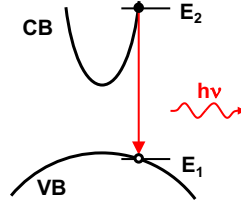
$$\alpha(E_{21})dE = \frac{r_{net}^{abs}(E_{21})dE}{P(E_{21})(c/n_r)} = \frac{n_r}{c} B[f_1 - f_2] = -g(E_{21})dE$$

$$\frac{r_{21}^{spont}(E_{21})}{g(E_{21})} = \frac{A_{21}}{\frac{n_r}{c} B} \left[ \frac{f_2(1-f_1)}{f_2 - f_1} \right] \leftarrow n_{sp}$$

$$r_{21}^{spont}(E_{21}) = \frac{8\pi n_r^2 E_{21}^2}{h^3 c^2} \left[ \frac{1}{1 - e^{-\frac{E_{21} - \Delta F}{k_B T}}} \right] g(E_{21}) \quad \left[ \frac{1}{s} \frac{1}{m^3} \frac{1}{eV} \right]$$

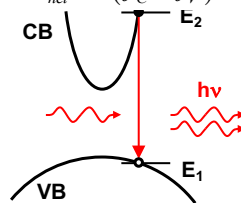
## Spontaneous Emission

$$R^{spont} \propto f_C(1-f_V)$$



## Stimulated Emission

$$R_{net}^{stim} \propto (f_C - f_V)$$

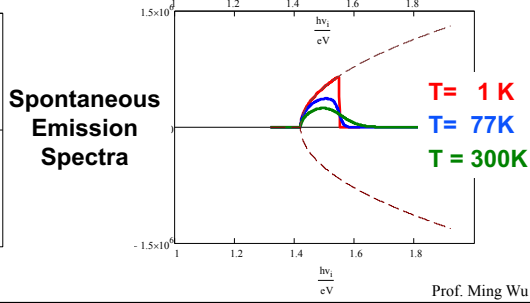
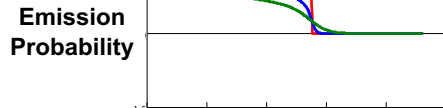
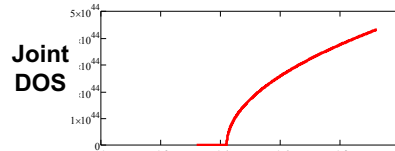
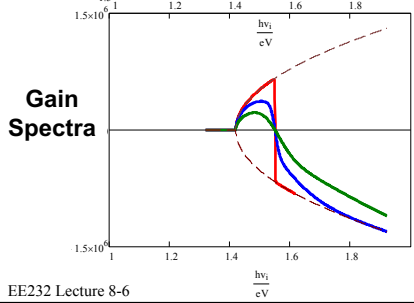
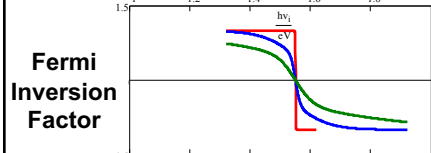
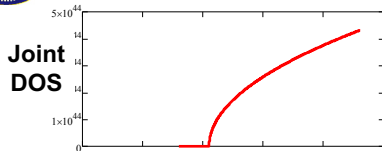


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# Spontaneous Emission and Gain Spectra for Various Temperatures

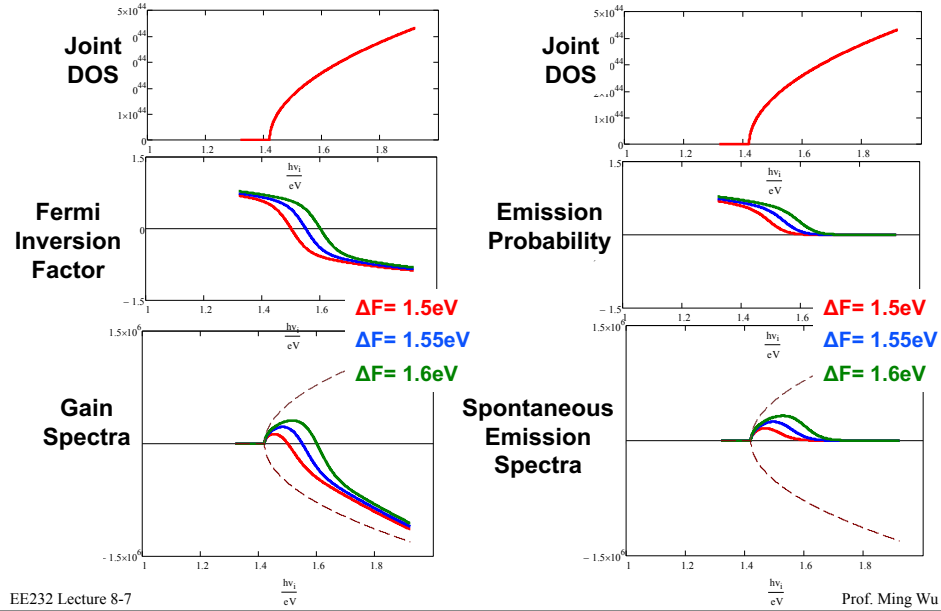


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## Spontaneous Emission and Gain Spectra for $\Delta F$ (T = 300 K)



## Spontaneous Emission Lifetime

$$r_{21}^{spon}(\hbar\omega) = \frac{1}{\tau_r} \rho_r(\hbar\omega - E_g) f_e(\hbar\omega)$$

$$f_e(\hbar\omega) = f_c(E_2) (1 - f_v(E_1))$$

$$r_{21}^{spon}(E_{21}) = \frac{8\pi n_r^2 E_{21}^2}{h^3 c^2} \frac{1}{1 - e^{-\frac{E_{21} - \Delta F}{k_B T}}} g(E_{21})$$

$$= \frac{8\pi n_r^2 E_{21}^2}{h^3 c^2} \frac{f_e(\hbar\omega)}{f_g(\hbar\omega)} \left( C_0 \left| \hat{e} \cdot \vec{P}_{cv} \right|^2 \rho_r(\hbar\omega - E_g) \right) f_g(\hbar\omega)$$

$$\Rightarrow \tau_r = \frac{h^3 c^2}{8\pi n_r^2 E_{21}^2} \cdot \frac{1}{C_0 \left| \hat{e} \cdot \vec{P}_{cv} \right|^2}$$

Typically  $\tau_r \sim 1$  nsec

$$\text{Flux\_per\_eV} := \frac{8 \cdot \pi \cdot n_r^2 \cdot E_g^2}{(2 \cdot \pi \cdot h\_bar)^3 \cdot c^2} \quad \text{Flux\_per\_eV} = 6.078 \times 10^{47} \frac{\text{s}}{\text{m}^4 \cdot \text{kg}}$$

$$\tau_r := \frac{1}{\text{Flux\_per\_eV}} \cdot \frac{1}{C_0 \frac{\text{m}^0}{6} \cdot E_p}$$

$$\tau_r = 5.443 \times 10^{-10} \text{ s}$$

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## Alternative Form for Gain

$$C_0 \left| \hat{e} \cdot \bar{P}_{cv} \right|^2 = \frac{h^3 c^2}{8\pi n_r^2 E_{21}^2} \cdot \frac{1}{\tau_r} = \frac{h\lambda_0^2}{8\pi n_r^2 \tau_r}$$
$$g(\hbar\omega) = \frac{h\lambda_0^2}{8\pi n_r^2 \tau_r} \rho_r(\hbar\omega - E_g) f_g(\hbar\omega)$$